Synthetic Division of Polynomials

Synthetic division is a shorthand, or shortcut, method of polynomial division in the special case of dividing by a linear factor -- and it *only* works in this case. Synthetic division is generally used, however, not for dividing out factors but for finding zeroes (or roots) of polynomials. More about this later.

If you are given, say, the polynomial equation $y = x^2 + 5x + 6$, you can factor the polynomial as $y = (x + 3)(x + 2)$. Then you can find the zeroes of y by setting each factor equal to zero and solving. You will find that $x = -2$ and $x = -3$ are the two zeroes of y.

You can, however, also work backwards from the zeroes to find the originating polynomial. For instance, if you are given that $x = -2$ and $x = -3$ are the zeroes of a quadratic, then you know that $x + 2 = 0$, so $x + 2$ is a factor, and $x + 3 = 0$, so $x + 3$ is a factor. Therefore, you know that the quadratic must be of the form $y = a(x + 3)(x + 2)$.

(The extra number "a" in that last sentence is in there because, when you are working backwards from the zeroes, you don't know toward which quadratic you're working. For any non-zero value of "a", your quadratic will still have the same zeroes. But the issue of the value of "a" is just a technical consideration; as long as you see the relationship between the zeroes and the factors, that's all you really need to know for this lesson.)

Anyway, the above is a long-winded way of saying that, if $x - n$ is a factor, then $x = n$ is a zero, and if $x = n$ is a zero, then $x - n$ is a factor. And this is the fact you use when you do synthetic division.

Let's look again at the quadratic from above: $y = x^2 + 5x + 6$. From the Rational Roots Test, you know that $\pm$ 1, 2, 3, and 6 are possible zeroes of the quadratic. (And, from the factoring above, you know that the zeroes are, in fact, -3 and -2.) How would you use synthetic division to check the potential zeroes? Well, think about how long polynomial division works. If we guess that $x = 1$ is a zero, then this means that $x - 1$ is a factor of the quadratic. And if it's a factor, then it will divide out evenly; that is, if we divide $x^2 + 5x + 6$ by $x - 1$, we would get a zero remainder. Let's check:

$$\begin{array}{r} x + 6 \\ x-1 \,\big)\overline{\, x^2 + 5x + 6} \\ \underline{x^2 - x} \\ 6x + 6 \\ \underline{6x - 6} \\ 12 \end{array}$$

As expected (since we know that $x - 1$ is not a factor), we got a non-zero remainder. What does this look like in synthetic division?

First, write the coefficients ONLY inside an upside-down division symbol:

$$\begin{array}{|ccc} 1 & 5 & 6 \\ & & \\ \hline \end{array}$$

Make sure you leave room inside, underneath the row of coefficients, to write another row of numbers later.

Put the test zero, $x = 1$, at the left:

1	1	5	6

Take the first number inside, representing the leading coefficient, and carry it down, unchanged, to below the division symbol:

1	1	5	6
	1		

Multiply this carry-down value by the test zero, and carry the result up into the next column:

1	1	5	6
		1	
	1		

Add down the column:

1	1	5	6
		1	
	1	6	

Multiply the previous carry-down value by the test zero, and carry the new result up into the last column:

1	1	5	6
		1	6
	1	6	

Add down the column:

This last carry-down value is the remainder.

1	1	5	6
		1	6
	1	6	12

Comparing, you can see that we got the same result from the synthetic division, the same quotient (namely, $1x + 6$) and the same remainder at the end (namely, 12), as when we did the long division:

$$\begin{array}{r|l} & x + 6 \\ \hline x-1 & x^2 + 5x + 6 \\ & \underline{x^2 - \ \ x} \\ & \quad 6x + 6 \\ & \quad \underline{6x - 6} \\ & \qquad 12 \end{array}$$

1	1	5	6
		1	6
	1	6	12

The results are formatted differently, but you should recognize that each format provided us with the result, being a quotient of $x + 6$, and a remainder of 12.

You already know (from the factoring above) that $x + 3$ is a factor of the polynomial, and therefore that

$x = -3$ is a zero. Now compare the results of long division and synthetic division when we use the factor $x + 3$ (for the long division) and the zero $x = -3$ (for the synthetic division):

$$x+3\overline{)\,x^2 + 5x + 6}$$

$$\begin{array}{r|rrr} -3 & 1 & 5 & 6 \\ & & & \\ \hline \end{array}$$

As you can see above, while the results are formatted differently, the results are otherwise the same:

In the long division, I divided by the factor $x + 3$, and arrived at the result of $x + 2$ with a remainder of zero. This means that $x + 3$ is a factor, and that $x + 2$ is left after factoring out the $x + 3$. Setting the factors equal to zero, I get that $x = -3$ and $x = -2$ are the zeroes of the quadratic.

In the synthetic division, I divided by $x = -3$, and arrived at the same result of $x + 2$ with a remainder of zero. Because the remainder is zero, this means that $x + 3$ is a factor and $x = -3$ is a zero. Also, because of the zero remainder, $x + 2$ is the remaining factor after division. Setting this equal to zero, I get that $x = -2$ is the other zero of the quadratic.

I will return to this relationship between factors and zeroes throughout what follows; the two topics are inextricably intertwined.

- **Complete the indicated division.**

$$\begin{array}{r|rrrrr} 2 & 2 & -3 & -5 & 3 & 8 \\ & & & & & \\ \hline \end{array}$$

For this first exercise, I will display the entire synthetic-division process step-by-step.

First, carry down the "2" that indicates the leading coefficient:

$$\begin{array}{r|rrrrr} 2 & 2 & -3 & -5 & 3 & 8 \\ & \downarrow & & & & \\ \hline & 2 & & & & \end{array}$$

Multiply by the number on the left, and carry the result into the next column:

$$\begin{array}{r|rrrrr} 2 & 2 & -3 & -5 & 3 & 8 \\ & & 4 & & & \\ \hline & 2 & & & & \end{array}$$

Add down the column:

$$\begin{array}{r|rrrrr} 2 & 2 & -3 & -5 & 3 & 8 \\ & & 4 & & & \\ \hline & 2 & 1 & & & \end{array}$$

Multiply by the number on the left, and carry the result into the next column:

2	2	−3	−5	3	8
		4	2		
	2	1			

Add down the column:

2	2	−3	−5	3	8
		4	2		
	2	1	−3		

Multiply by the number on the left, and carry the result into the next column:

2	2	−3	−5	3	8
		4	2	−6	
	2	1	−3		

Add down the column:

2	2	−3	−5	3	8
		4	2	−6	
	2	1	−3	−3	

Multiply by the number on the left, and carry the result into the next column:

2	2	−3	−5	3	8
		4	2	−6	−6
	2	1	−3	−3	

Add down the column for the remainder:

2	2	−3	−5	3	8
		4	2	−6	−6
	2	1	−3	−3	2

The completed division is:

2	2	−3	−5	3	8
		4	2	−6	−6
	2	1	−3	−3	2

This exercise never said anything about polynomials, factors, or zeroes, but this division says that, if you divide $2x^4 – 3x^3 – 5x^2 + 3x + 8$ by $x – 2$, then the remainder will be 2, and therefore $x – 2$ is not a factor of $2x^4 – 3x^3 – 5x^2 + 3x + 8$, and $x = 2$ is not a zero (that is, a root or x-intercept) of the initial polynomial.

- **Divide $3x^3 – 2x^2 + 3x – 4$ by $x – 3$ using synthetic division. Write the answer in the form " $q(x) + {}^{r(x)}/_{d(x)}$ ".**

  This question is asking me, in effect, to convert an "improper" polynomial "fraction" into a polynomial "mixed number". That is, I am being asked to do something similar to converting the improper fraction $^{17}/_5$ to the mixed number $3\ ^2/_5$, which is really the shorthand for the addition expression "$3 + {}^2/_5$".

To convert the polynomial division into the required "mixed number" format, I have to do the division; I will show most of the steps.

First, write down all the coefficients, and put the zero from $x - 3 = 0$ (so $x = 3$) at the left.

$$\begin{array}{r|rrrr} 3 & 3 & -2 & 3 & -4 \\ & & & & \\ \hline & & & & \end{array}$$

Next, carry down the leading coefficient:

$$\begin{array}{r|rrrr} 3 & 3 & -2 & 3 & -4 \\ & & & & \\ \hline & 3 & & & \end{array}$$

Multiply by the potential zero, carry up to the next column, and add down:

$$\begin{array}{r|rrrr} 3 & 3 & -2 & 3 & -4 \\ & & 9 & & \\ \hline & 3 & 7 & & \end{array}$$

Repeat this process:

$$\begin{array}{r|rrrr} 3 & 3 & -2 & 3 & -4 \\ & & 9 & 21 & \\ \hline & 3 & 7 & 24 & \end{array}$$

Repeat this process again:

$$\begin{array}{r|rrrr} 3 & 3 & -2 & 3 & -4 \\ & & 9 & 21 & 72 \\ \hline & 3 & 7 & 24 & 68 \end{array}$$

As you can see, the remainder is 68. Since I started with a polynomial of degree 3 and then divided by $x - 3$ (that is, by a polynomial of degree 1), I am left with a polynomial of degree 2. Then the bottom line represents the polynomial $3x^2 + 7x + 24$ with a remainder of 68. Putting this result into the required "mixed number" format, I get the answer as being:

$$3x^2 + 7x + 24 + \frac{68}{x-3}$$

It is always true that, when you use synthetic division, your answer (in the bottom row) will be of degree one less than what you'd started with, because you have divided out a linear factor. That was how I knew that my answer, denoted by the "$3\ \ 7\ \ 24$" in the bottom row, stood for a quadratic, since I had started with a cubic.

Once you know how to do synthetic division, you can use the technique as a shortcut to finding factors and zeroes of polynomials. Here are some examples:

- **Use synthetic division to determine whether $x = 1$ is a zero of $x^3 – 1$.**

  Set up the synthetic division, and check to see if the remainder is zero. If the remainder is zero, then $x = 1$ is a zero of $x^3 – 1$.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & & & \\ \hline & & & & \end{array}$$

  To do the initial set-up, note that I needed to leave "gaps" for the powers of x that are not included in the polynomial. That is, I followed the practice used with long division, and wrote the polynomial as $x^3 + 0x^2 + 0x – 1$ for the purposes of doing the division. If you forget to leave "gaps", your division will not work properly!

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & & & \\ \hline & 1 & & & \end{array}$$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & 1 & & \\ \hline & 1 & 1 & & \end{array}$$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & 1 & 1 & \\ \hline & 1 & 1 & 1 & \end{array}$$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

  Since the remainder is zero, then **$x = 1$ is a zero of $x^3 – 1$**.

Since $x = 1$ is a zero of $x^3 – 1$, then $x – 1$ is a factor, so the polynomial $x^3 – 1$ factors as $(x – 1)(x^2 + x + 1)$.

- **Use synthetic division to determine whether $x – 4$ is a factor of $–2x^5 + 6x^4 + 10x^3 – 6x^2 – 9x + 4$.**

  For $x – 4$ to be a factor, you must have $x = 4$ as a zero. Using this information, I'll do the synthetic division with $x = 4$ as the test zero on the left:

$$\begin{array}{r|rrrrrr} 4 & -2 & 6 & 10 & -6 & -9 & 4 \\ & & -8 & -8 & 8 & 8 & -4 \\ \hline & -2 & -2 & 2 & 2 & -1 & 0 \end{array}$$

Since the remainder is zero, then $x = 4$ is indeed a zero of $-2x^5 + 6x^4 + 10x^3 - 6x^2 - 9x + 4$, so:

Yes, $x - 4$ is a factor of $-2x^5 + 6x^4 + 10x^3 - 6x^2 - 9x + 4$

- **Find all the factors of $15x^4 + x^3 - 52x^2 + 20x + 16$ by using synthetic division.**

Remember that, if $x = a$ is a zero, then $x - a$ is a factor. So use the Rational Roots Test (and maybe a quick graph) to find a good value to test for a zero (x-intercept). I'll try $x = 1$:

$$\begin{array}{r|rrrrr} 1 & 15 & 1 & -52 & 20 & 16 \\ & & 15 & 16 & -36 & -16 \\ \hline & 15 & 16 & -36 & -16 & 0 \end{array}$$

This division gives a zero remainder, so $x = 1$ must be a zero, which means that $x - 1$ is a factor. Since I divided a linear factor (namely, $x - 1$) out of the original polynomial, then my result has to be a cubic: $15x^3 + 16x^2 - 36x - 16$. So I need to find another zero before I can apply the Quadratic Formula. I'll try $x = -2$:

$$\begin{array}{r|rrrrr} 1 & 15 & 1 & -52 & 20 & 16 \\ & & 15 & 16 & -36 & -16 \\ \hline -2 & 15 & 16 & -36 & -16 & 0 \\ & & -30 & 28 & 16 & \\ \hline & 15 & -14 & -8 & 0 & \end{array}$$

Since I got a zero remainder, then $x = -2$ is a zero, so $x + 2$ is a factor. Plus, I'm now down to a quadratic, $15x^2 - 14x - 8$, which happens to factor as:

$$(3x - 4)(5x + 2)$$

Then the fully-factored form of the original polynomial is:

$$15x^4 + x^3 - 52x^2 + 20x + 16$$

$$= \mathbf{(x - 1)(x + 2)(3x - 4)(5x + 2)}$$

Practice

1. $(x^3+3x^2 - 5x + 6) \div (x - 3)$
2. $(2x^4 - 5x^3+6x^2 - 10) \div (x + 4)$
3. Is $(x - 2)$ a factor of $(x^4-4x^3+2x^2 - 6)$?