Long Division of Polynomials

If you're dividing a polynomial by something more complicated than just a simple monomial, then you'll need to use a different method for the simplification. That method is called "long (polynomial) division", and it works just like the long (numerical) division you did back in elementary school, except that now you're dividing with variables.

- **Divide $x^2 - 9x - 10$ by $x + 1$**

  Think back to when you were doing long division with plain old numbers. You would be given one number that you had to divide into another number. You set up the division symbol, inserted the two numbers where they belonged, and then started making guesses. And you didn't guess the whole answer right away; instead, you started working on the "front" part (the larger place values) of the number you were dividing.

  Long division for polynomials works in much the same way:

First, I set up the division:

$$x+1\overline{)x^2 \ \ -9x-10}$$

For the moment, I'll ignore the other terms and look just at the leading x of the divisor and the leading x^2 of the dividend.

If I divide the leading x^2 inside by the leading x in front, what would I get? I'd get an x. So I'll put an x on top:

$$\begin{array}{r l} & \ \ x \\ x+1 & \overline{)x^2 \ \ -9x-10} \end{array}$$

Now I'll take that x, and multiply it through the divisor, $x + 1$. First, I multiply the x (on top) by the x (on the "side"), and carry the x^2 underneath:

$$\begin{array}{r l} & \ \ x \\ x+1 & \overline{)x^2 \ \ -9x-10} \\ & \ \ x^2 \end{array}$$

Then I'll multiply the x (on top) by the 1 (on the "side"), and carry the $1x$ underneath:

$$\begin{array}{r l} & \ \ x \\ x+1 & \overline{)x^2 \ \ -9x-10} \\ & \ \ x^2 \ \ +1x \end{array}$$

Then I'll draw the "equals" bar, so I can do the subtraction.

To subtract the polynomials, I *change all the signs* in the second line...

$$\begin{array}{r l} & \ \ x \\ x+1 & \overline{)x^2 \ \ -9x-10} \\ & \underline{-x^2 \ \ \overset{-}{+}1x \quad\quad} \end{array}$$

...and then I add down. The first term (the x^2) will cancel out:

$$\begin{array}{r l} & \ \ x \\ x+1 & \overline{)x^2 \ \ -9x-10} \\ & \underline{-x^2 \ \ \overset{-}{+}1x \quad\quad} \\ & \quad\quad -10x \end{array}$$

I need to remember to carry down that last term,
the "subtract ten", from the dividend:

$$\begin{array}{rl} & x \\ x+1 & \overline{)\, x^2 - 9x - 10} \\ & \underline{-x^2 \overset{-}{+} 1x } \\ & -10x - 10 \end{array}$$

Now I look at the x from the divisor and the new leading term, the $-10x$, in the bottom line of the division. If I divide the $-10x$ by the x, I would end up with a -10, so I'll put that on top:

$$\begin{array}{rl} & x - 10 \\ x+1 & \overline{)\, x^2 - 9x - 10} \\ & \underline{-x^2 \overset{-}{+} 1x } \\ & -10x - 10 \end{array}$$

Now I'll multiply the -10 (on top) by the leading x (on the "side"), and carry the $-10x$ to the bottom:

$$\begin{array}{rl} & x - 10 \\ x+1 & \overline{)\, x^2 - 9x - 10} \\ & \underline{-x^2 \overset{-}{+} 1x } \\ & -10x - 10 \\ & -10x \end{array}$$

...and I'll multiply the -10 (on top) by the 1 (on the "side"), and carry the -10 to the bottom:

$$\begin{array}{rl} & x - 10 \\ x+1 & \overline{)\, x^2 - 9x - 10} \\ & \underline{-x^2 \overset{-}{+} 1x } \\ & -10x - 10 \\ & -10x - 10 \end{array}$$

I draw the equals bar, and *change the signs* on all the terms in the bottom row:

$$\begin{array}{rl} & x - 10 \\ x+1 & \overline{)\, x^2 - 9x - 10} \\ & \underline{-x^2 \overset{-}{+} 1x } \\ & -10x - 10 \\ & \underline{+10x + 10} \end{array}$$

Then I add down:

Then the solution to this division is: (***x* – 10**)

$$\begin{array}{rl} & x - 10 \\ x+1 & \overline{)\, x^2 - 9x - 10} \\ & \underline{-x^2 \overset{-}{+} 1x } \\ & -10x - 10 \\ & \underline{+10x + 10} \\ & 0 \end{array}$$

Since the remainder on this division was zero (that is, since there wasn't anything left over), the division came out "even". When you do regular division with numbers and the division comes out even, it means that the number you divided by is a factor of the number you're dividing. For instance, if you divide 50 by 10, the answer will be a nice neat "5" with a zero remainder, because 10 is a factor of 50. In the case of the above polynomial division, the zero remainder tells us that $x + 1$ is a factor of $x^2 - 9x - 10$, which you can confirm by factoring the original quadratic dividend, $x^2 - 9x - 10$.

- **Simplify** $\dfrac{x^2 + 9x + 14}{x + 7}$

  This can be done in either of two ways: I can factor the quadratic and then cancel the common factor, like this:

  $$\frac{x^2 + 9x + 14}{x + 7} = \frac{(x+2)(x+7)}{x+7} = x + 2$$

  But what if I didn't know how to factor?

  I can always use long division: (I mustn't forget to change my signs, as shown in red, when I'm doing the subtraction.)

  $$\begin{array}{r|l} & x + 2 \\ \hline x+7 & x^2 + 9x + 14 \\ & -x^2 - 7x \\ \hline & \quad\;\; 2x + 14 \\ & \quad -2x - 14 \\ \hline & \qquad\quad 0 \end{array}$$

  The answer to the division is quotient, the polynomial across the top: ***x* + 2**

- **Divide $3x^3 - 5x^2 + 10x - 3$ by $3x + 1$**

$$\begin{array}{r|l} & x^2 - 2x + 4 \\ \hline 3x+1 & 3x^3 - 5x^2 + 10x - 3 \\ & -3x^3 - 1x^2 \\ \hline & \quad -6x^2 + 10x - 3 \\ & \quad +6x^2 + 2x \\ \hline & \qquad\quad 12x - 3 \\ & \qquad -12x - 4 \\ \hline & \qquad\qquad -7 \end{array}$$

This division did not come out even.

What am I supposed to do with the remainder?

Think back to when you did long division with plain numbers.

Sometimes there would be a remainder; for instance, if you divide 132 by 5:

$$\begin{array}{r|l} & \;\;26 \\ \hline 5 & 132 \\ & 10 \\ \hline & \;\;32 \\ & \;\;30 \\ \hline & \;\;\;2 \end{array}$$

...there is a remainder of 2. Remember how you handled that?

You made a fraction, putting the remainder on top of the divisor, and wrote the answer as "twenty-six and two-fifths":

$$132 \div 5 = 26\frac{2}{5} = 26 + \frac{2}{5}$$

The first form, without the "plus" in the middle, is how "mixed numbers" are written, but the meaning of the mixed number is actually the addition.

We do the same thing with polynomial division. Since the remainder is -7 and since the divisor is $3x + 1$, then I'll turn the remainder into a fraction (the remainder divided by the original divisor), and add this fraction to the polynomial across the top of the division symbol. Since the division looks like this:

$$\begin{array}{r|rrrr}
 & x^2 & -2x & +4 & \\
\hline
3x+1 & 3x^3 & -5x^2 & +10x & -3 \\
 & -3x^3 & -1x^2 & & \\
\hline
 & & -6x^2 & +10x & -3 \\
 & & +6x^2 & +2x & \\
\hline
 & & & 12x & -3 \\
 & & & -12x & -4 \\
\hline
 & & & & -7
\end{array}$$

...then the answer is this:

$$x^2 - 2x + 4 + \frac{-7}{3x+1}$$

Warning: Do *not* write the polynomial "mixed number" in the same format as numerical mixed numbers! If you just append the fractional part to the polynomial part, this will be interpreted as polynomial multiplication, which is *not* what you mean!

- **Divide $2x^3 - 9x^2 + 15$ by $2x - 5$**

  First off, I note that there is a gap in the degrees of the terms of the dividend: the polynomial $2x^3 - 9x^2 + 15$ has no x term. My work could get very messy inside the division symbol, so it is important that I leave space for a x-term column, just in case. I can create this space by turning the dividend into $2x^3 - 9x^2 + 0x + 15$. This is a legitimate mathematical step: since I've only added zero, I haven't actually changed the value of anything. Now that I have all the "room" I might need for my work, I'll do the division:

$$\begin{array}{r|rrrr}
 & x^2 & -2x & -5 & \\
\hline
2x-5 & 2x^3 & -9x^2 & +0x & +15 \\
 & -2x^3 & +5x^2 & & \\
\hline
 & & -4x^2 & +0x & +15 \\
 & & +4x^2 & -10x & \\
\hline
 & & & -10x & +15 \\
 & & & +10x & -25 \\
\hline
 & & & & -10
\end{array}$$

$$x^2 - 2x - 5 + \frac{-10}{2x-5}$$

I need to remember to *add* the remainder to the polynomial part of the answer:

- **Divide $4x^4 + 3x^3 + 2x + 1$ by $x^2 + x + 2$**

  I'll add a $0x^2$ term to the dividend (inside the division symbol) to make space for my work, and then I'll do the division in the usual manner:

$$
\begin{array}{r|l}
 & 4x^2 \quad -x \quad -7 \\
x^2+x+2 & 4x^4+3x^3+0x^2+2x+1 \\
 & -4x^4-4x^3-8x^2 \\
\hline
 & -x^3-8x^2+2x+1 \\
 & +x^3+x^2+2x \\
\hline
 & -7x^2+4x+1 \\
 & +7x^2+7x+14 \\
\hline
 & 11x+15
\end{array}
$$

  Then my answer is: $4x^2 - x - 7 + \dfrac{11x+15}{x^2+x+2}$

To succeed with polynomial long division, you need to write neatly, remember to change your signs when you're subtracting, and work carefully, keeping your columns lined up properly. If you do this, then these exercises should not be very hard; tedious, maybe, but not hard.

Practice

1. $(x^3 - 4x^2 + 2x - 5) \div (x + 3)$
2. $(3x^4 + 5x^3 - 2x^2 + 3) \div (3x - 2)$
3. $(x^3 - 8) \div (x - 2)$
4. $(2x^4 - 3x^3 + 4x^2 - 6x + 3) \div (x^2 + x + 1)$