Graph of the Trigonometric Functions

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Abstract

This handout discusses the graph of the six trigonometric functions, their properties and transformations (translations and stretching) of these graphs.

1 Graphs of the Trigonometric Functions and their Properties

We begin with some general definitions before studying each trigonometric function. As we have noted before, given an angle θ, the angles $\theta + 2k\pi$ are coterminal and therefore determine the same point on the unit circle. It follows that $\sin(\theta) = \sin(\theta + 2k\pi)$. The same is true for the cosine function. This means that these functions will repeat themselves. When a function has this property, it is called periodic. More precisely, we have the following definition:

Definition 1 (periodic function) *A function $y = f(x)$ is called **periodic** if there exists a positive constant p such that $f(x+p) = f(x)$ for any x in the domain of f. The smallest such number p is called the **period** of the function.*

Remark 2 *This number p mentioned in the definition is not unique. If p works, so will any multiple of p For example if $f(x+p) = f(x)$ then $f(x+2p) = f(x)$ because $f(x+2p) = f(x+p+p) = f(x+p) = f(x)$. The period of a function is the smallest of all the numbers p such that $f(x+p) = f(x)$.*

Remark 3 *One can think of the period as the length of the shortest interval over which the function repeats itself. Once we know the values of a periodic function over an interval having the length of its period, then we know the values of the function over its entire domain. This means that when we study a periodic function, we only need to study it on an interval having the length of its period.*

Definition 4 (amplitude) *The amplitude of a periodic function $y = f(x)$ is defined to be one half the distance between its maximum value and its minimum value.*

Remark 5 (notation) *When dealing with trigonometric functions, we break some of the notation rules we usually follow with functions. Here are some examples:*

1. *With functions in general, the input values are always in parentheses. The function f evaluated at x is denoted $f(x)$. For trigonometric functions, when the input is a single symbol such as a number or a variable, we omit the parentheses. However, if the input is an expression containing more than one symbol, we* ***must*** *use the parentheses. Thus, we write:*
   - $\sin x$
   - $\sin \pi$
   - $\cos 120$
   - $\sin(x+\pi)$
   - $\cos(\alpha+\beta)$
2. *If we want to raise a trigonometric function to a power n, we should write $(\sin x)^n$ or $(\cos(x+\pi))^n$. The same being true for the other trigonometric functions. However, instead, we write $\sin^n x$, $\cos^n(x+\pi)$.*

We are now ready to study each trigonometric function. For each function, we will study the following:

1. domain
2. range
3. period
4. amplitude
5. graph
6. additional properties

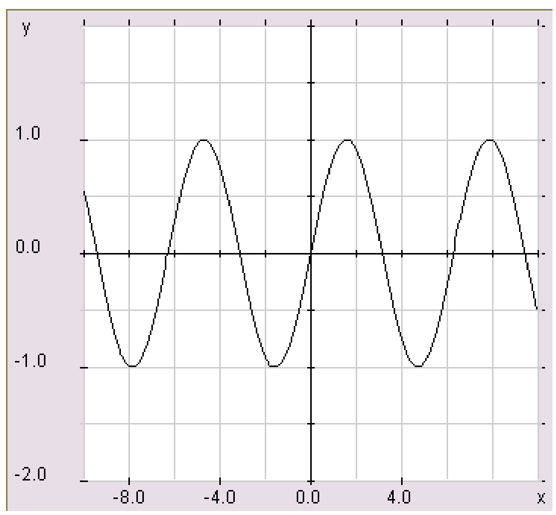
1.1 The Sine Function: $y = \sin x$

- **Domain**: The domain of the sine function is the set of real numbers. To each θ (think of θ as an angle) corresponds a point on the unit circle. Its $y-coordinate$ is $\sin\theta$.
- **Range:** To each angle θ corresponds a point on the unit circle. The coordinates of this point are $(\cos\theta, \sin\theta)$. Also, the coordinates of a point on the unit circle are numbers between -1 and 1. This means that $-1 \leq \sin\theta \leq 1$. Thus, the range of $\sin x$ is $[-1, 1]$.
- **Period:** The period of $\sin x$ is 2π. This means that this function repeats itself every interval of length 2π.

- **Amplitude:** By definition, we have

$$\begin{aligned} amplitude &= \frac{\max - \min}{2} \\ &= \frac{1-(-1)}{2} \\ &= 1 \end{aligned}$$

- **Graph:** To help visualize the graph, students can use an applet which can be found at http://science.kennesaw.edu/~plaval/applets/SinCosDef.html. Using this applet, students can move a point along the unit circle. As the point moves, the graph of either the sine or the cosine function is traced. Students are strongly encouraged to use this applet to understand why the graph of $y = \sin x$ looks the way it does. The shape of the graph of sine is given by the figure below.



One can see in particular that the sine function repeats itself.

- **Additional Properties:**

  - $y = \sin x$ is an odd function, that is

$$\sin(-x) = -\sin x$$

1.2 The Cosine Function: $y = \cos x$

- **Domain**: The domain of the cosine function is the set of real numbers. To each θ (think of θ as an angle) corresponds a point on the unit circle. Its $x - coordinate$ is $\cos\theta$.

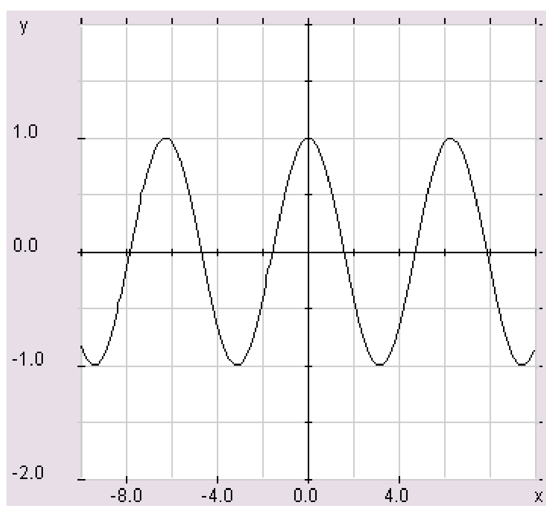
- **Range:** To each angle θ corresponds a point on the unit circle. The coordinates of this point are $(\cos\theta, \sin\theta)$. Also, the coordinates of a point on the unit circle are numbers between -1 and 1. This means that $-1 \leq \cos\theta \leq 1$. Thus, the range of $\cos x$ is $[-1, 1]$.

- **Period:** The period of $\cos x$ is 2π. This means that this function repeats itself every interval of length 2π.

- **Amplitude:** By definition, we have

$$\begin{aligned} amplitude &= \frac{\max - \min}{2} \\ &= \frac{1 - (-1)}{2} \\ &= 1 \end{aligned}$$

- **Graph:** To help visualize the graph, students can use an applet which can be found at http://science.kennesaw.edu/~plaval/applets/SinCosDef.html. Using this applet, students can move a point along the unit circle. As the point moves, the graph of either the sine or the cosine function is traced. Students are strongly encouraged to use this applet to understand why the graph of $y = \cos x$ looks the way it does. The shape of the graph of sine is given by the figure below.



- **Additional Properties:**

  - $y = \cos x$ is an even function, that is

$$\cos(-x) = \cos x$$

1.3 Summary for $\sin x$ and $\cos x$

	$y = \sin x$	$y = \cos x$
Domain	$\mathbb{R}$	$\mathbb{R}$
Range	$[-1, 1]$	$[-1, 1]$
Period	2π	2π
Amplitude	1	1
Properties	$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$

1.4 The Tangent Function: $y = \tan x$

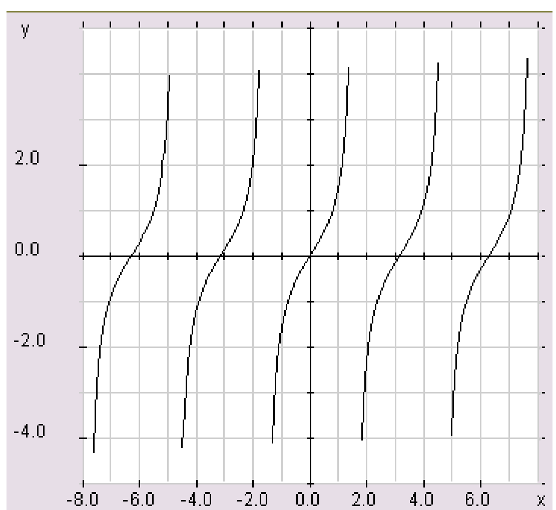
- **Domain:** By definition, $\tan x = \dfrac{\sin x}{\cos x}$. This means that $\tan x$ is not defined whenever $\cos x = 0$. This happens when $x = \dfrac{\pi}{2} + k\pi$, where k is any integer.

- **Range:** The range of $\tan x$ is $\mathbb{R}$.

- **Period:** The period of $\tan x$ is π.

- **Amplitude:** There is no amplitude since $\tan x$ has no maximum or minimum.

- **Graph:** The graph is shown below:



  One can see that the tangent function repeats itself every interval of length π. At the points where $\tan x$ is not defined, its values get arbitrary large, meaning that they are approaching $\pm\infty$.

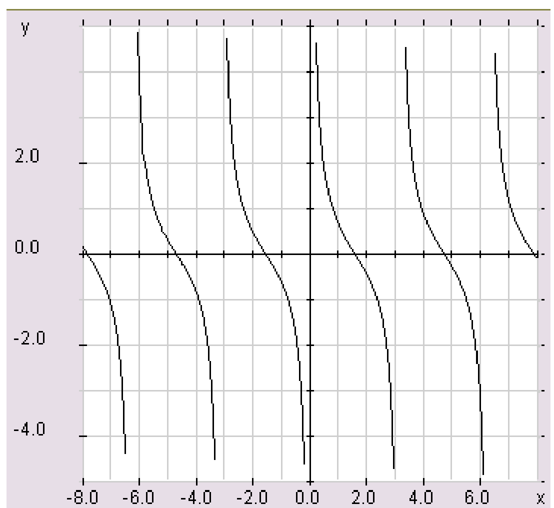
- **Additional Properties:**

– $y = \tan x$ is an odd function, meaning that

$$\tan(-x) = -\tan x$$

1.5 The Cotangent Function $y = \cot x$

- **Domain:** By definition, $\cot x = \dfrac{\cos x}{\sin x}$. This means that $\cot x$ is not defined whenever $\sin x = 0$. This happens when $x = k\pi$, where k is any integer.
- **Range:** The range of $\cot x$ is $\mathbb{R}$.
- **Period:** The period of $\cot x$ is π.
- **Amplitude:** There is no amplitude since $\cot x$ has no maximum or minimum.
- **Graph:** The graph is shown below:



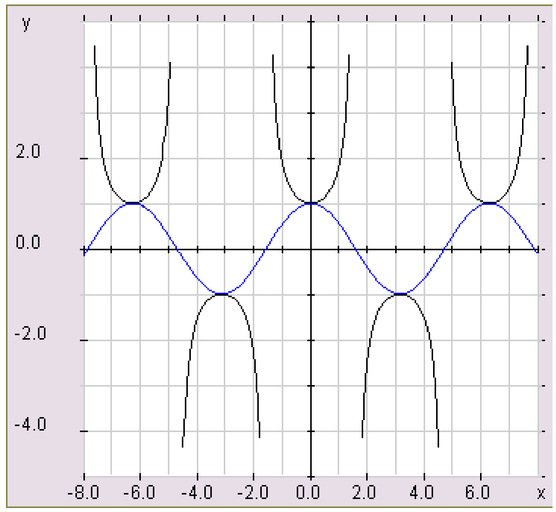
- **Additional Properties:**
  – $y = \cot x$ is an odd function, meaning that

$$\cot(-x) = -\cot x$$

1.6 The Secant Function $y = \sec x$

- **Domain:** By definition, $\sec x = \dfrac{1}{\cos x}$. This means that $\sec x$ is not defined whenever $\cos x = 0$. This happens when $x = \dfrac{\pi}{2} + k\pi$, where k is any integer.

- **Range:** The range of $\sec x$ is $(-\infty, -1] \cup [1, \infty)$.
- **Period:** The period of $\sec x$ is 2π.
- **Amplitude:** There is no amplitude since $\sec x$ has no maximum or minimum.
- **Graph:** The graph is shown below:



  Note that this shows the graph of two functions. $y = \sec x$ is in black, $y = \cos x$ is in blue. The two functions are shown simply to illustrate how they are related.

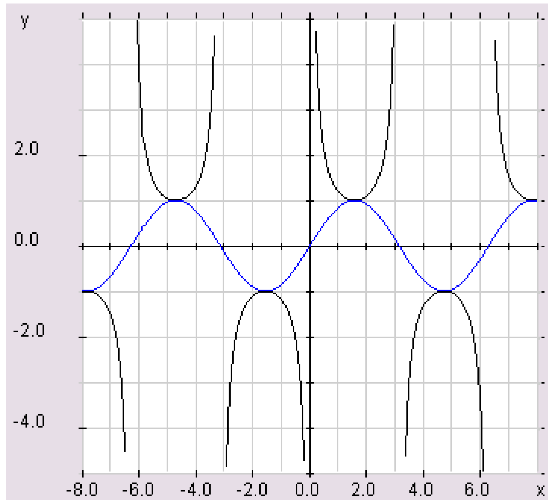
- **Additional Properties:**
  - $y = \sec x$ is an even function, meaning that

$$\sec(-x) = \sec x$$

1.7 The Cosecant Functions $y = \csc x$

- **Domain:** By definition, $\csc x = \dfrac{1}{\sin x}$. This means that $\csc x$ is not defined whenever $\sin x = 0$. This happens when $x = k\pi$, where k is any integer.
- **Range:** The range of $\csc x$ is $(-\infty, -1] \cup [1, \infty)$.
- **Period:** The period of $\csc x$ is 2π.
- **Amplitude:** There is no amplitude since $\csc x$ has no maximum or minimum.

- **Graph:** The graph is shown below:



Note that this shows the graph of two functions. $y = \csc x$ is in black, $y = \sin x$ is in blue. The two functions are shown simply to illustrate how they are related.

- **Additional Properties:**
  - $y = \csc x$ is an odd function, meaning that

$$\csc(-x) = -\csc x$$

1.8 Summary for $\tan x$, $\cot x$, $\sec x$ and $\csc x$

	$y = \tan x$	$y = \cot x$	$y = \sec x$	$y = \csc x$
Domain	$\mathbb{R} - \left\{\frac{\pi}{2} + k\pi\right\}$	$\mathbb{R} - \{k\pi\}$	$\mathbb{R} - \left\{\frac{\pi}{2} + k\pi\right\}$	$\mathbb{R} - \{k\pi\}$
Range	$\mathbb{R}$	$\mathbb{R}$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
Period	π	π	2π	2π
Amplitude	none	none	none	none
Properties	$\tan(-x) = -\tan x$	$\cot(-x) = -\cot x$	$\sec(-x) = \sec x$	$\csc(-x) = -\csc x$

2 Transformations of the Graphs of $\sin x$ and $\cos x$

In this section, we look at the graphs of $y = a + b\sin(k(x-c))$ and $y = a + b\cos(k(x-c))$ by treating them, as transformations of the graphs of $y = \sin x$ and $y = \cos x$. You will recall that there are 4 kinds of transformations which are:

- $x \to x - c$ produces a horizontal shift of $|c|$ units to the right if $c > 0$ and to the left if $c < 0$.
- $y \to y - a$ produces a vertical shift of $|a|$ units up if $a > 0$ and down if $a < 0$.
- $x \to kx$ produces a horizontal shrinking by a factor of $\frac{1}{k}$if $|k| < 1$ and a horizontal stretching by a factor of $\frac{1}{k}$ if $0 < |k| < 1$. If k is also negative, the graph is also reflected about the $y - axis$.
- $y \to by$ produces a vertical shrinking by a factor of $\frac{1}{b}$if $|b| < 1$ and a vertical stretching by a factor of $\frac{1}{b}$ if $0 < |b| < 1$. If b is also negative, the graph is also reflected about the $x - axis$.

2.1 Graph of $y = a + b\sin(k(x - c))$ or $y = a + b\cos(k(x - c))$

These can be obtained from $y = \sin x$ or $y = \cos x$ by applying the following transformations in the given order. Keep in mind that each transformation is applied to the function we obtained from the previous transformation. For each transformation, we look at which of the attributes (period, interval over which the function repeats itself, amplitude, range) are changed and which are not. We explain the process below using $\sin x$. The results are the same for $\cos x$.

1. $x \to kx$. The resulting function is $y = \sin kx$. The graph is either shrunk or stretched horizontally by a factor of $\frac{1}{|k|}$. This means that the interval over which the function repeats itself will have a width of $\frac{2\pi}{|k|}$. Therefore, the period of the function will be $\frac{2\pi}{|k|}$. The amplitude or the range of the function will not be affected since there are $y - values$. If in addition $k < 0$, the graph will be reflected across the $y - axis$.
2. $x \to x - c$. The resulting function is $y = \sin(k(x - c))$. The graph is shifted $|c|$ units. The period, interval over which the function repeats itself, amplitude and range are not changed. The shift of the graph is called **phase shift**.
3. $y \to \frac{1}{b}y$. The resulting function is $\frac{1}{b}y = \sin(k(x - c))$ or $y = b\sin(k(x - c))$. The graph is shrunk or stretched vertically by a factor of $|b|$. Therefore the period and the interval over which the function repeats itself are not changed (these are $x - values$). The range is changed to $[-|b|, |b|]$, the amplitude is $|b|$. If in addition $b < 0$, the graph will be reflected across the $x - axis$.

4. $y \to y - a$. The resulting function is $y - a = b\sin(k(x-c))$ or $y = a + b\sin(k(x-c))$. The graph is shifted vertically by $|a|$ units. The period, interval over which the function repeats itself, amplitude and range are not changed (these are $x - values$). The amplitude is not changed either an is still $|b|$ since the shape of the graph is preserved, we only shift it. The range will become $[-|b| + a, |b| + a]$.

Visit the internet, on the page which contains all the applets I have developed for my classes. Under the heading "Trigonometric functions, use the applet entitled "Transformation of sine". This applet will allow the user to experiment with the function $y = a + b\sin(k(x-c))$ by changing the values of a, b, c and k and watch the graph change.

2.2 Summary

To graph $y = a + b\sin(k(x-c))$ or $y = a + b\cos(k(x-c))$, start from the graph of $y = \sin x$ or $y = \cos x$, then follow the steps below:

1. Stretch or shrink the graph horizontally according to k by a factor of $\dfrac{1}{|k|}$ as follows:
   - If $|k| < 1$, stretch the graph horizontally.
   - If $|k| > 1$, shrink the graph horizontally.
   - If in addition $k < 0$, reflect the graph across the $y - axis$.
2. Translate the graph horizontally by $|c|$ units as follows:
   - To the right if $c > 0$.
   - To the left if $c < 0$.
3. Stretch or shrink the graph vertically according to b by a factor of $\dfrac{1}{|b|}$ as follows:
   - If $|b| < 1$, shrink the graph vertically.
   - If $|b| > 1$, stretch the graph vertically.
   - If in addition $b < 0$, reflect the graph across the $x - axis$.
4. Translate the graph vertically by $|a|$ units as follows:
   (a) Up if $a > 0$.
   (b) Down if $a < 0$.

The attributes (period, range, amplitude, phase shift) are affected by these transformation as follows:

1. **Period:** The period of $y = a + b\sin(k(x-c))$ or $y = a + b\cos(k(x-c))$ is $\dfrac{2\pi}{|k|}$. So, it is only affected by k.

2. Range: The range of $y = a + b\sin(k(x-c))$ or $y = a + b\cos(k(x-c))$ is $[-|b| + a, |b| + a]$. So, it is affected by a and b.

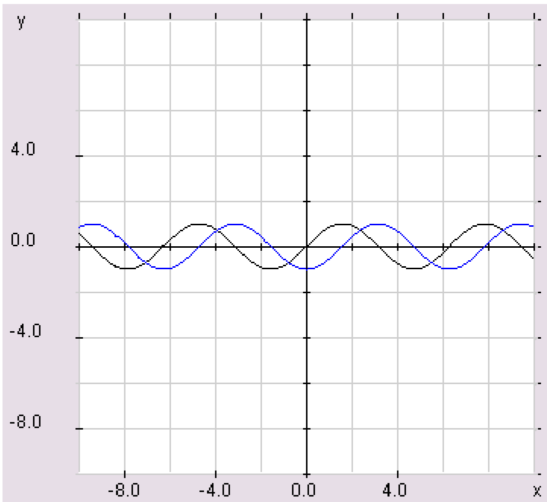
3. Amplitude: The amplitude of $y = a+b\sin(k(x-c))$ or $y = a+b\cos(k(x-c))$ is $|b|$. So, it is affected by b.

4. Phase shift: The phase shift of $y = a + b\sin(k(x-c))$ or $y = a + b\cos(k(x-c))$ c. It is only affected by c.

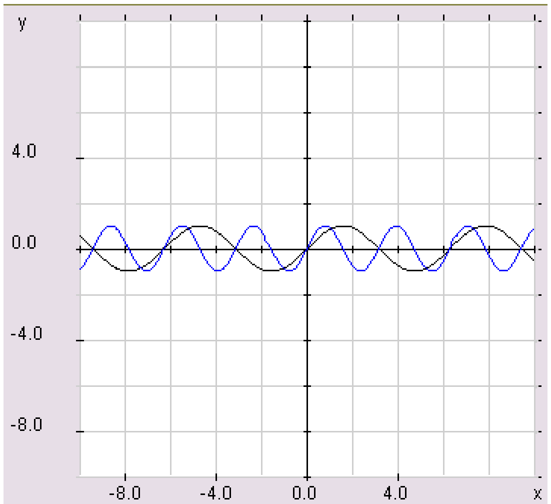
We illustrate this by first looking at examples which involve only one of the above transformations at a time. We then look at more general examples involving several transformations.

Example 6 *Find the period, amplitude, range and phase shift of* $y = \sin\left(x - \dfrac{\pi}{2}\right)$. *Describe how its graph can be obtained from the graph of* $y = \sin x$ *then sketch its graph.*

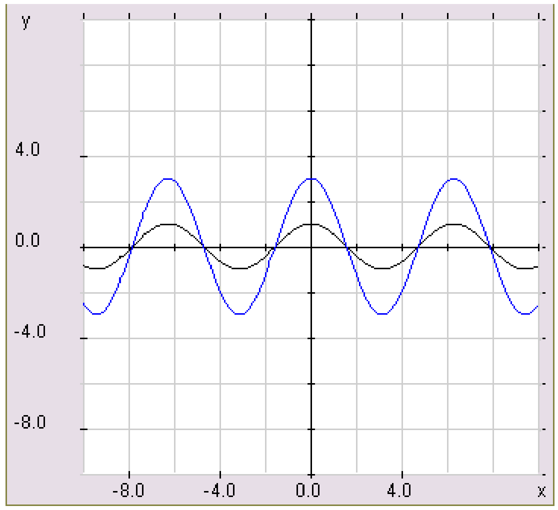
There is only one transformation involved here: $x \to x - \dfrac{\pi}{2}$*. Using the notation of the explanation above,* $c = \dfrac{\pi}{2}$, $a = 0$, $b = k = 1$*. Thus, the phase shift is* $\dfrac{\pi}{2}$*, the period is unchanged and is* 2π*, the range is unchanged and is* $[-1, 1]$*, the amplitude is unchanged and is* 1*. This function is obtained by shifting the graph of* $y = \sin x$ $\dfrac{\pi}{2}$ *units to the right. The graphs of* $\sin x$ *and* $\sin\left(x - \dfrac{\pi}{2}\right)$ *are shown below.* $y = \sin x$ *is in black,* $y = \sin\left(x - \dfrac{\pi}{2}\right)$ *is in blue.*



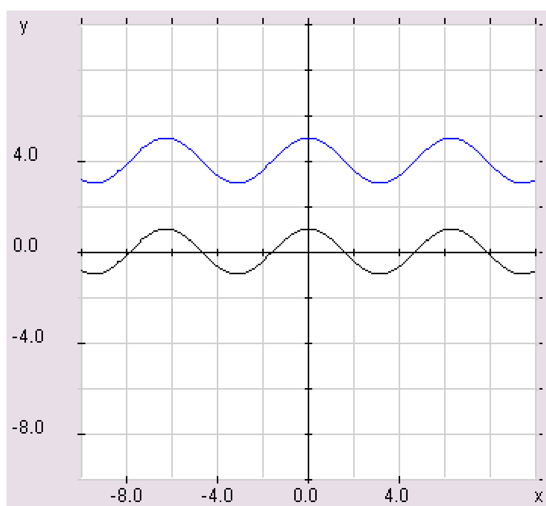
Example 7 *Find the period, amplitude, range and phase shift of* $y = \sin 2x$. *Describe how its graph can be obtained from the graph of* $y = \sin x$ *then sketch its graph.*
There is only one transformation which is $x \to 2x$. *This is a horizontal shrinking. To use the notation of the explanations above, we have* $a = c = 0$, $b = 1$ *and* $k = 2$. *Therefore, the period will be* $\frac{2\pi}{2} = \pi$. *The other quantities will remain unchanged. The range is still* $[1, 1]$, *the amplitude is still* 1, *the phase shift is* 0. *The graphs of* $\sin x$ *and* $\sin(2x)$ *are shown below.* $y = \sin x$ *is in black,* $y = \sin(2x)$ *is in blue.*



Example 8 *Find the period, amplitude, range and phase shift of* $y = 3\cos x$. *Describe how its graph can be obtained from the graph of* $y = \cos x$ *then sketch its graph.*
Using the notation above, we have $a = c = 0$, $k = 1$ *and* $b = 3$. *Therefore, the period is unchanged and is* 2π, *the phase shift is* 0, *the range is* $[-3, 3]$ *and the amplitude is* 3. *The graph is obtained by stretching the graph of* $\cos x$ *by a factor of* 3. *The graphs of* $\cos x$ *and* $3\cos x$ *are shown below.* $y = \cos x$ *is in black,* $y = 3\cos x$ *is in blue.*

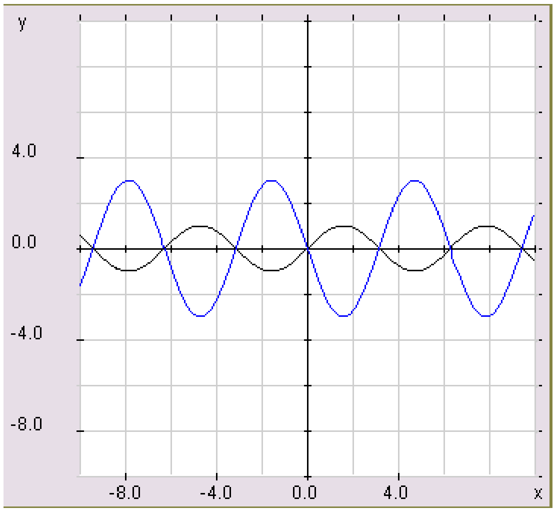


Example 9 *Find the period, amplitude, range and phase shift of* $y = 4 + \cos x$. *Describe how its graph can be obtained from the graph of* $y = \cos x$ *then sketch its graph.*
Using the notation above, we have $a = 4$, $c = 0$, $k = 1$ *and* $b = 1$. *Therefore, the period is unchanged and is* 2π, *the phase shift is* 0, *the range is* $[-1 + 4, 1 + 4] = [3, 5]$ *and the amplitude is* 1. *The graph is obtained by shifting the graph of* $\cos x$ 4 *units up. The graphs of* $\cos x$ *and* $4 + \cos x$ *are shown below.* $y = \cos x$ *is in black,* $y = 4 + \cos x$ *is in blue.*

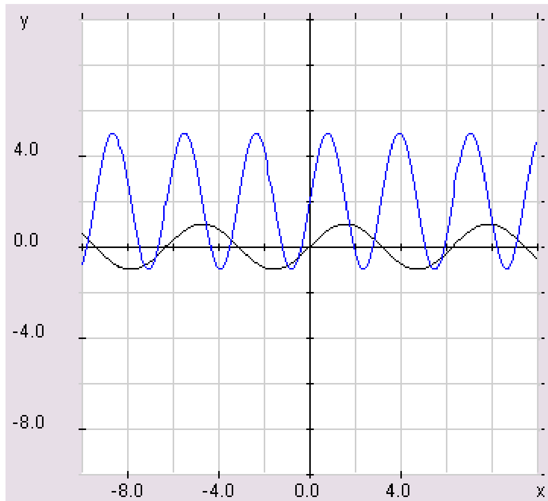


Example 10 *Find the period, amplitude, range and phase shift of* $y = 3\sin(x - \pi)$. *Describe how its graph can be obtained from the graph of* $y = \sin x$ *then sketch*

its graph.
Using the notation of the explanations above, we have $a = 0$*,* $b = 3$*,* $k = 1$ *and* $c = \pi$*. Therefore, the phase shift is* π*, the period is unchanged and is* 2π*. The range is* $[-3, 3]$ *and the amplitude is* 3*. The graph of* $y = 3\sin(x - \pi)$ *is obtained by shifting the graph of* $y = \sin x$ π *units to the right, then stretching it vertically by a factor of* 3*. The graphs of* $\sin x$ *and* $3\sin(x - \pi)$ *are shown below.* $y = \sin x$ *is in black,* $y = 3\sin(x - \pi)$ *is in blue.*



Example 11 *Find the period, amplitude, range and phase shift of* $y = 2 + 3\sin(2(x - \pi))$*. Describe how its graph can be obtained from the graph of* $y = \sin x$ *then sketch its graph.*
Using the notation of the explanations above, we have $a = 2$*,* $b = 3$*,* $k = 2$*, and* $c = \pi$*. Therefore, the period of this function is* $\frac{2\pi}{2} = \pi$*. The range is* $[-3 + 2, 3 + 2] = [-1, 5]$*. The amplitude is* 3*. The phase shift is* π*. The graph of* $y = 2 + 3\sin(2(x - \pi))$ *is obtained by shifting the graph of* $y = \sin x$ π *units to the right, shrinking it horizontally by a factor of* 2*, stretching it vertically by a factor of* 3 *and shifting it up* 3 *units. The graphs of* $\sin x$ *and* $2 + 3\sin(2(x - \pi))$ *are shown below.* $y = \sin x$ *is in black,* $y = 2 + 3\sin(2(x - \pi))$ *is in blue.*



3 Problems

The above explanations can also be found in your book in sections 5.5 and 5.6. The reader should be careful though as the notation used in the book is slightly different. The notation used in the notes is consistent with the notation of the various applets which can be used to illustrate the topics explained. The reader should be able to do the following problems:

1. From section 5.5, # 43, 44, 45, 49, 50, 52, 53, 54 on pages 480, 481.

2. From section 5.6, # 1, 2, 3, 9, 10, 14, 16, 18, 19, 20, 23, 24 on page 497.